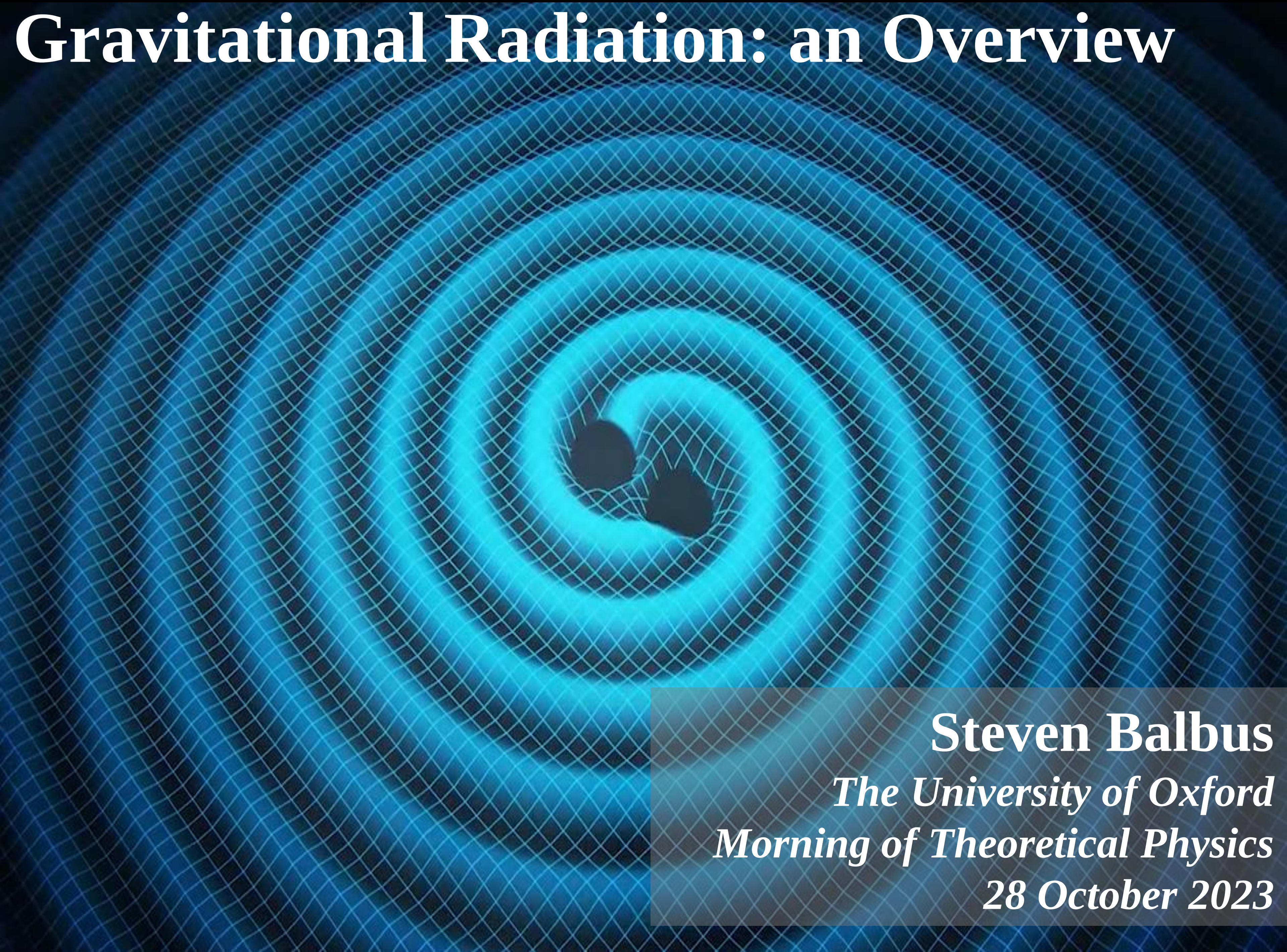
Gravitational Radiation: an Overview



Steven Balbus

The University of Oxford

Morning of Theoretical Physics

28 October 2023

The Languages of Gravity

Three languages for gravity:

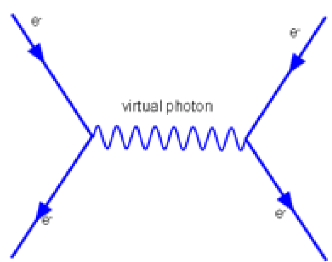
$$\Phi = -G \int \frac{\rho(\mathbf{r}')d^3r'}{|\mathbf{r} - \mathbf{r}'|}$$

(Isaac Newton)

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = -8\pi G T_{\mu\nu}$$

(Albert Einstein)

$$W = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{*\mu\nu} \frac{(g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda}) - g_{\mu\nu}g_{\lambda\sigma}}{k^2 + i\epsilon} T^{\lambda\sigma}$$



(Richard Feynman)

Three languages for gravity:

Incorporate causality

$$\Phi = -G \int \frac{\rho(\mathbf{r}')d^3r'}{|\mathbf{r}-\mathbf{r}'|}$$ (Isaac Newton)

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = -8\pi G T_{\mu\nu}$$ (Albert Einstein)

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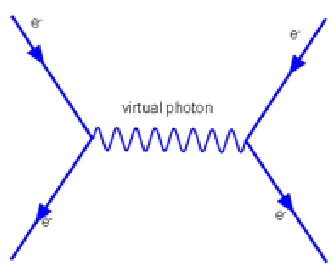


(Richard Feynman)

Three languages for gravity:

$$\Phi = -G \int \frac{\rho(\mathbf{r}')d^3r'}{|\mathbf{r} - \mathbf{r}'|}$$

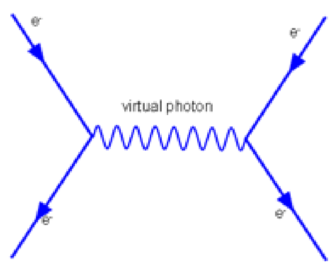
(Isaac Newton)

Incorporate QM

$$R_{\mu\nu} - \frac{g_{\mu\nu}}{2}R = -8\pi G T_{\mu\nu}$$

(Albert Einstein)

$$W = -\frac{1}{2}\int \frac{d^4k}{(2\pi)^4} T^{*\mu\nu} \frac{(g_{\mu\lambda}g_{\nu\sigma} + g_{\mu\sigma}g_{\nu\lambda}) - g_{\mu\nu}g_{\lambda\sigma}}{k^2 + i\epsilon} T^{\lambda\sigma}$$



(Richard Feynman)

Electromagnetism also comes in 3 languages:

1) Coulombian (Electrostatic Potential Theory),

2) Maxwellian (Classical Field Theory), and

3) Feynmanian (spin 1 photons and propagators)

We understand how 1, 2, and 3 combine fairly seamlessly.

Gravity, today is spoken in 3 languages:

1) Laplacian (Potential theory), $\boldsymbol{g} = -\nabla\Phi$

2) Riemannian (space-time manifolds), and

3) Feynmanian (spin 2 gravitons, propagators).

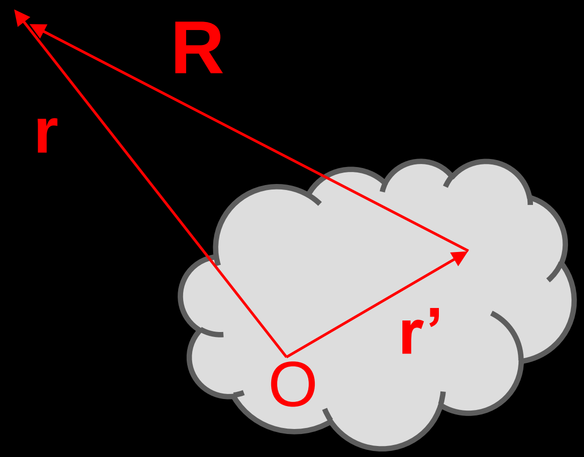
We understand how 1 and 2 combine, but not yet 3. Understanding gravitational radiation is essential for making gravity consistent with quantum mechanics.

$$\nabla^2 \Phi = 4\pi G \rho$$

Static potential theory

$$\Phi(\boldsymbol{r}) = -G \int \frac{\rho(\boldsymbol{r'}) d^3 r'}{R}$$

$$R = |\boldsymbol{r} - \boldsymbol{r'}|$$

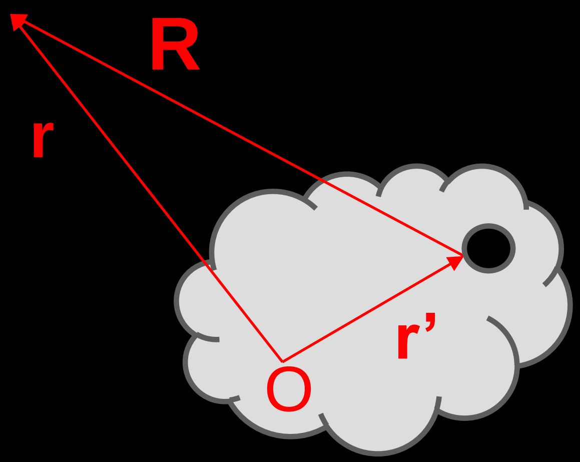


$$\nabla^2 \Phi = 4\pi G \rho$$

Time dependent theory?

$$\Phi(\boldsymbol{r}, t) = -G \int \frac{\rho(\boldsymbol{r'}, t) d^3 r'}{R}$$

$$R = |\boldsymbol{r} - \boldsymbol{r'}|$$



This is not what actually happens. We can see what happens by following Maxwell:

$$-\frac{1}{c^2}\frac{\partial^2\Phi_{EM}}{\partial t^2}+\nabla^2\Phi_{EM}=-4\pi\rho_{EM}$$

$$\Phi_{EM}(\boldsymbol{r},t)=\int\frac{\rho_{EM}(\boldsymbol{r'},t')d^3r'}{R}$$

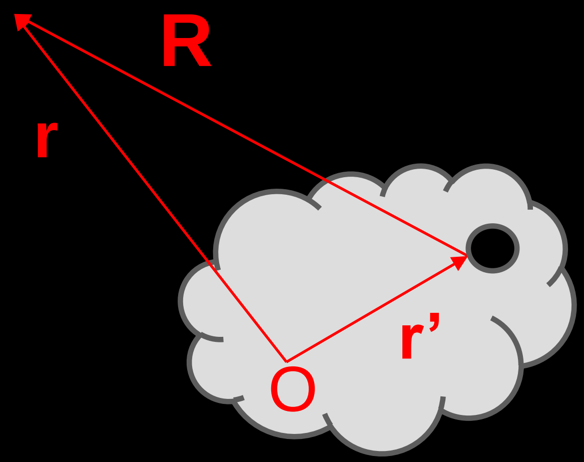
$$R=|\boldsymbol{r}-\boldsymbol{r'}|$$

$$t'=t-R/c$$

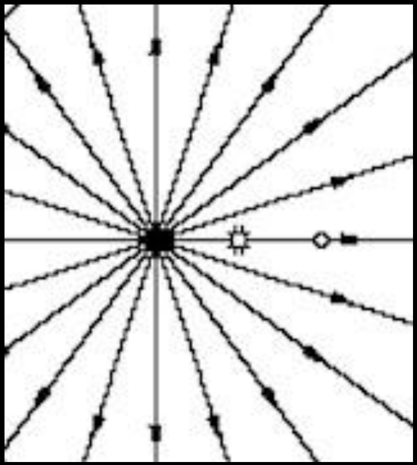
$$\Phi_{EM}(\boldsymbol{r}, t) = \int \frac{\rho_{EM}(\boldsymbol{r'}, t')d^3r'}{R}$$

$$R = |\boldsymbol{r} - \boldsymbol{r'}|$$

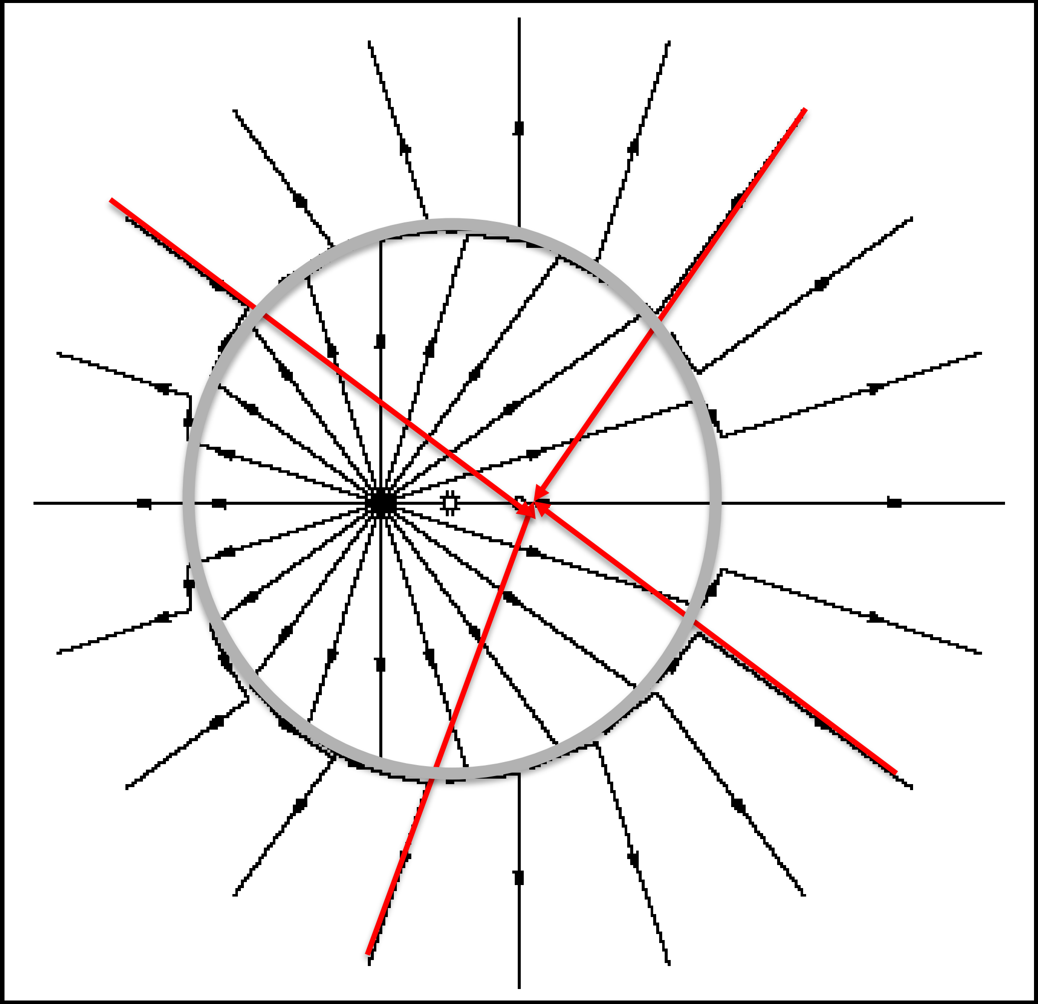
$$t' = t - R/c$$

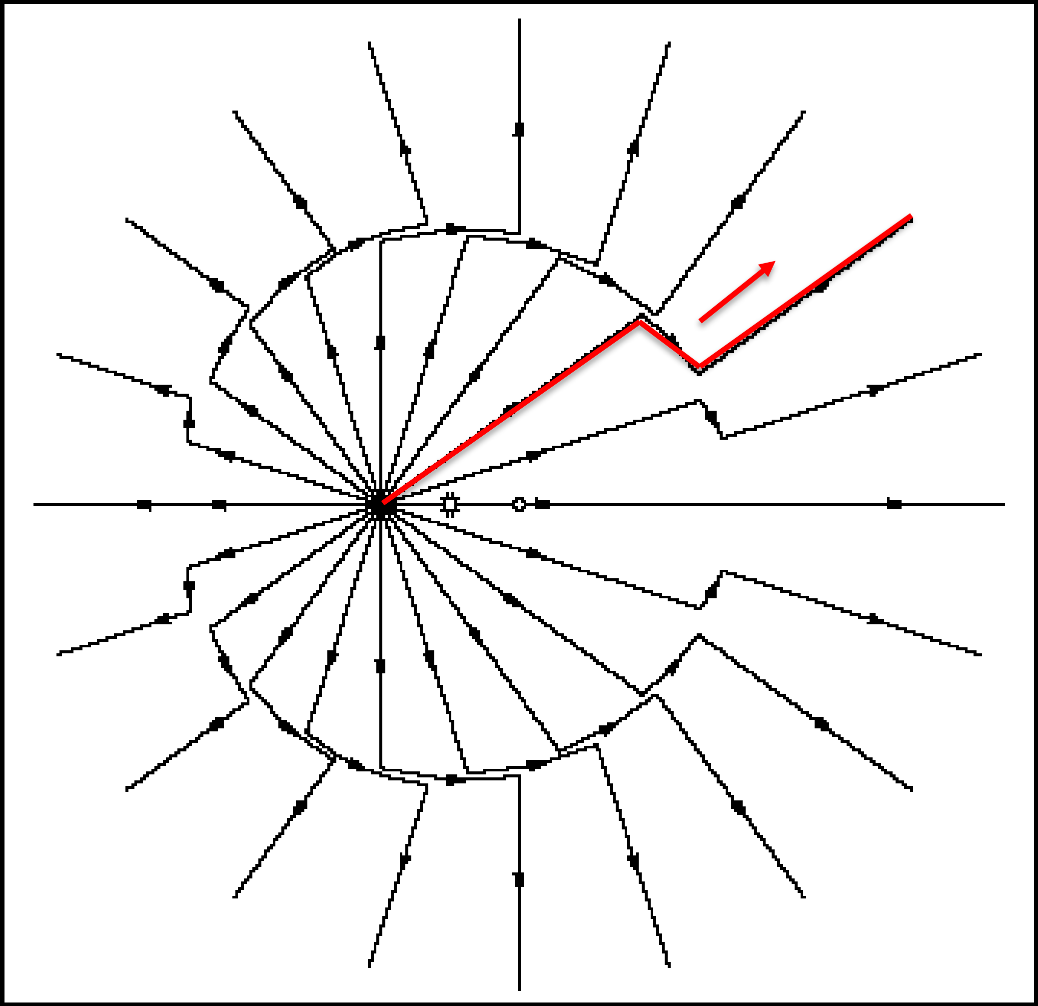


The charge density at **r’** at a time R/c before the current time t.



This is the direction of the lines of electrical force near a + charge.





Kink moves out at the speed of light!

Gravitational radiation is very similar.

In Einstein’s theory of gravity, we live in a space, a “Minkowski space” in which the interval, or separation between events, is given by $-dw^2 + dx^2 + dy^2 + dz^2$.

The odd sign dimension w is what our consciousness experiences as “time.”

dw is normally huge compared with dx, dy, dz so we write dw= c dt, where c is a very large number, the speed of light.

In Einstein's theory of gravity, gravity itself is not a force, but a distortion or "curvature" of how we measure this interval, or "metric". Here is the simplest metric for a black hole, found by Karl Schwarzschild in 1916:

$$-c^2\, dt^2\ (1-2GM/rc^2) + dr^2 / (1-2GM/rc^2) + d\Omega^2\ .$$

Demanding that the difference between the time part and the space part is a minimum, gives the equations of motion for orbits!

The world of special relativity:

$$-c^2 d\tau^2 = -c^2\, dt^2 + dx^2 + dy^2 + dz^2$$

$$= \eta_{\alpha\beta}\, dx^\alpha dx^\beta$$

With $dx^0 = c\,dt$, $dx^1 = dx$, $dx^2 = dy$, $dx^3 = dz$

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

More generally, when matter is present,

$$-c^2 d\tau^2 = g_{\alpha\beta} \; dx^\alpha dx^\beta$$

Where now $g_{\alpha\beta}$, the metric tensor, can be *anything,* depending upon coordinates and the curvature of the spactime.

$g_{\alpha\beta} = \eta_{\alpha\beta}$ is the world of special relativity.

$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, where all $h_{\alpha\beta} << 1$, is the world of gravitational radiation we will study.

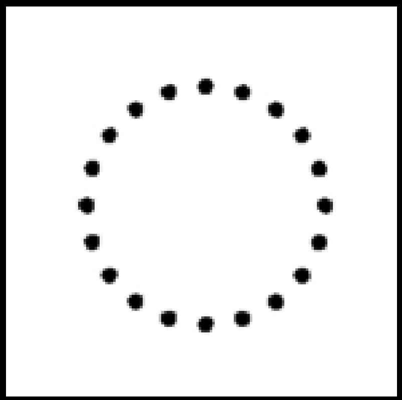
$$-\frac{1}{c^2}\frac{\partial^2 h_{\alpha\beta}}{\partial t^2} + \nabla^2 h_{\alpha\beta} = \Box h_{\alpha\beta} = 0$$

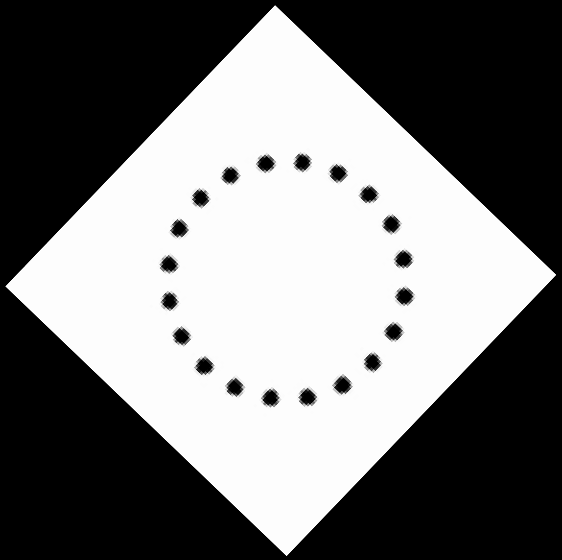
Away from the sources, the $h_{\alpha\beta}$ satisfy the same wave equation as the electric potential in Maxwell's theory!

There are two distinct modes of propagation. With $h_{\alpha\beta} = A_{\alpha\beta} \cos(kz - \omega t)$, $\omega = kc$,

1.) $h_{xx} = - h_{yy}$ (and nothing else)
2.) $h_{xy} = h_{yx}$ (and nothing else)

Two modes of GW polarisation





+ mode : $h_{xx} = - h_{yy}$ × mode: $h_{xy} = h_{yx}$

Note: interval dx^i goes to $dx^i + (1/2)\, h_{ij}\, dx^j$,
so dx goes to $dx + (1/2)(h_{xx}\, dx + h_{xy}\, dy)$

$$h_{\alpha\beta} = \frac{A_{\alpha\beta}}{r}\cos(kr - \omega t)$$

A radial gravitational radiation wave form.

Now, the + polarisation is $h_{\theta\theta} = -h_{\varphi\varphi}$, and the × polarization is $h_{\theta\varphi} = h_{\varphi\theta}$.

The ENERGY in a gravitational wave:
how hard is it to bend space and time?

$$-\frac{1}{c^2}\frac{\partial^2 h_{\alpha\beta}}{\partial t^2} + \nabla^2 h_{\alpha\beta} = \Box h_{\alpha\beta} = 0$$

Multiply by - $\partial h_{\alpha\beta}$ / ∂t and sum over α and β.

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\frac{1}{c^2}\frac{\partial h_{\alpha\beta}}{\partial t}\frac{\partial h_{\alpha\beta}}{\partial t} + (\boldsymbol{\nabla} h_{\alpha\beta})\cdot\boldsymbol{\nabla} h_{\alpha\beta}\right)$$

$$-\boldsymbol{\nabla}\cdot\left(\frac{\partial h_{\alpha\beta}}{\partial t}\boldsymbol{\nabla} h_{\alpha\beta}\right) = 0$$

The ENERGY a gravitational wave:
how hard is it to bend space and time?

$$\frac{\partial}{\partial t}\frac{1}{2}\left(\frac{1}{c^2}\frac{\partial h_{\alpha\beta}}{\partial t}\frac{\partial h_{\alpha\beta}}{\partial t} + (\boldsymbol{\nabla} h_{\alpha\beta})\cdot\boldsymbol{\nabla} h_{\alpha\beta})\right) - \boldsymbol{\nabla}\cdot\left(\frac{\partial h_{\alpha\beta}}{\partial t}\boldsymbol{\nabla} h_{\alpha\beta}\right) = 0$$

Has the form of any energy conservation equation: ∂/∂t (density) + div(flux) = 0.
But only up to an overall multiplicative factor!
Need to include the sources to determine that factor: rate at which grav. field does work on the sources!

Energy Density:

$$\frac{c^4}{64\pi G}\left(\frac{1}{c^2}\frac{\partial h_{\alpha\beta}}{\partial t}\frac{\partial h_{\alpha\beta}}{\partial t} + (\boldsymbol{\nabla} h_{\alpha\beta})\cdot\boldsymbol{\nabla} h_{\alpha\beta})\right)$$

Energy Flux:

$$-\frac{c^4}{32\pi G}\left(\frac{\partial h_{\alpha\beta}}{\partial t}\boldsymbol{\nabla} h_{\alpha\beta}\right)$$

Flux = Energy density × c for a plane wave.

For a sum of plane waves, energy density:

$$\rho_{GW}\, c^2 = \frac{c^2}{32\pi G} \left\langle \frac{\partial h_{\alpha\beta}}{\partial t} \frac{\partial h_{\alpha\beta}}{\partial t} \right\rangle$$

$$\left\langle \frac{\partial h_{\alpha\beta}}{\partial t} \frac{\partial h_{\alpha\beta}}{\partial t} \right\rangle = 8\pi^2 \int_0^\infty df\, f^2\, S_h(f)$$

S_h (f) is the "power spectral density" of a superposition of waves.

For a sum of plane waves, energy density:

$$\rho_{GW}\ c^2 = \frac{c^2}{32\pi G}\left\langle \frac{\partial h_{\alpha\beta}}{\partial t}\frac{\partial h_{\alpha\beta}}{\partial t}\right\rangle$$

becomes:

$$\rho_{GW}\ c^2 = \frac{\pi c^2}{4G}\int_0^\infty df\ f^2\ S_h(f)$$

Cosmologists like to study the background of gravitational waves. The critical energy density that would keep space exactly flat is given by

$$\rho_{crit}\, c^2 = \frac{3 H_0^2 c^2}{8\pi G}$$

$$\Omega_{GW}(f) = \frac{f}{\rho_{crit}} \frac{d\rho_{GW}}{df} = \frac{2\pi^2 f^3 S_h(f)}{3 H_0^2}$$

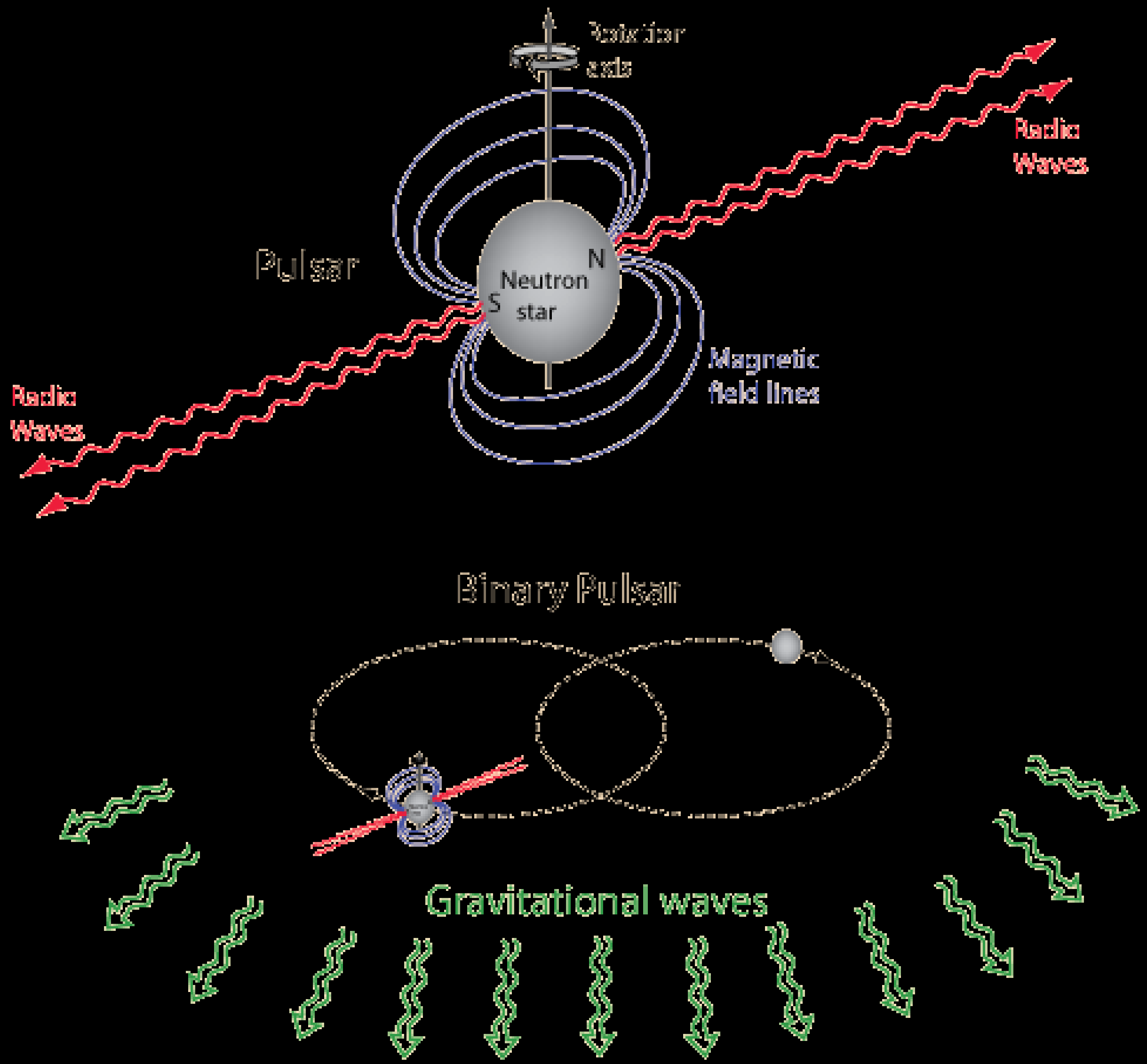
$$I_{ij} = \int x_i x_j \rho \, d^3x$$

$$J_{ij} = I_{ij} - I_{kk}\delta_{ij}/3$$

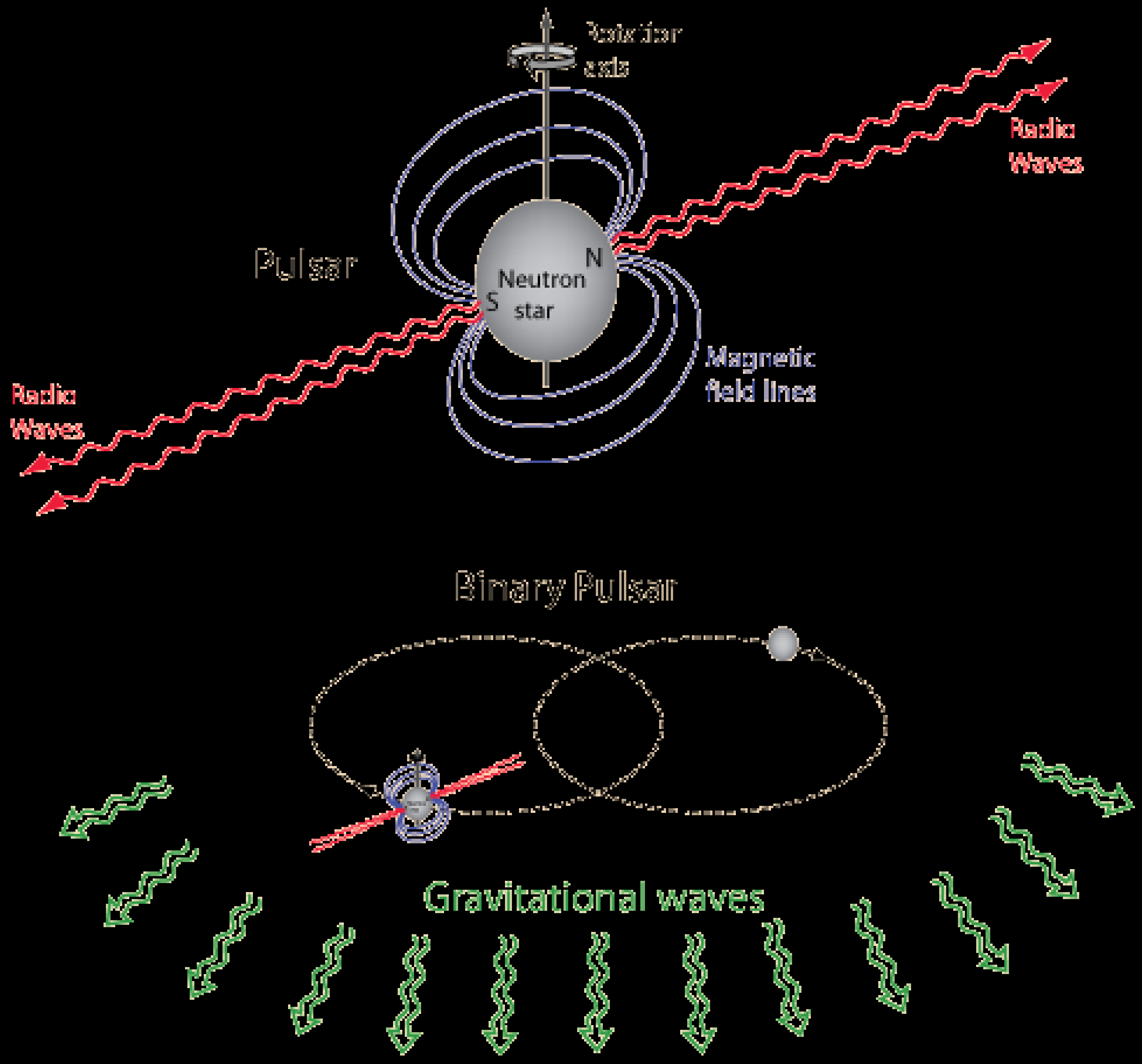
$$h_{ij} = \frac{2G}{c^4 r}\ddot{J}_{ij}$$

$$\dot{E}_{GW} = \frac{G}{5c^5}\dddot{J}_{ij}\dddot{J}_{ij}$$

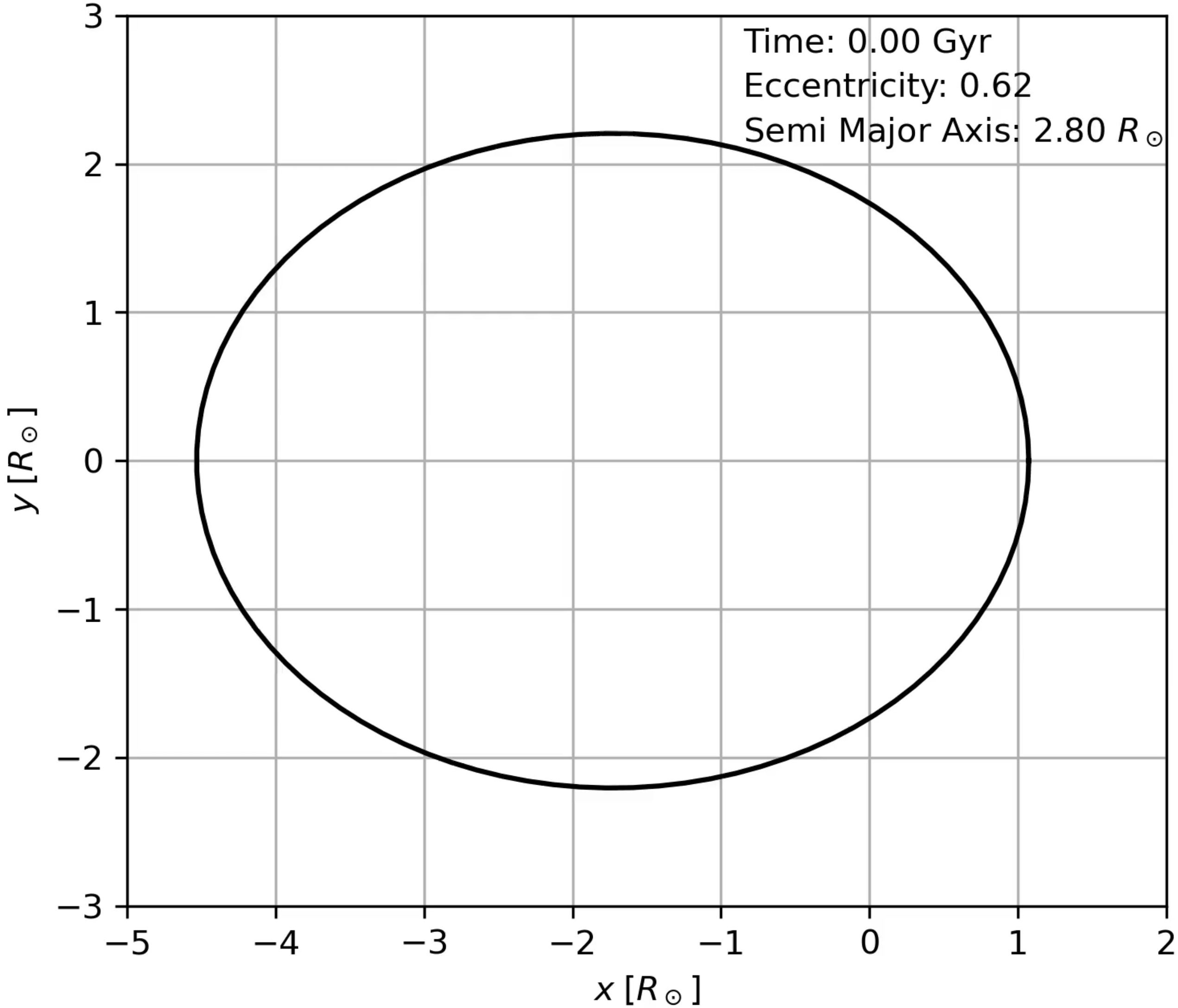
$$\dot{L}_{GW} = \frac{2G}{5c^5}\epsilon^{imk}\ddot{J}_{in}\dddot{J}_{mn}$$

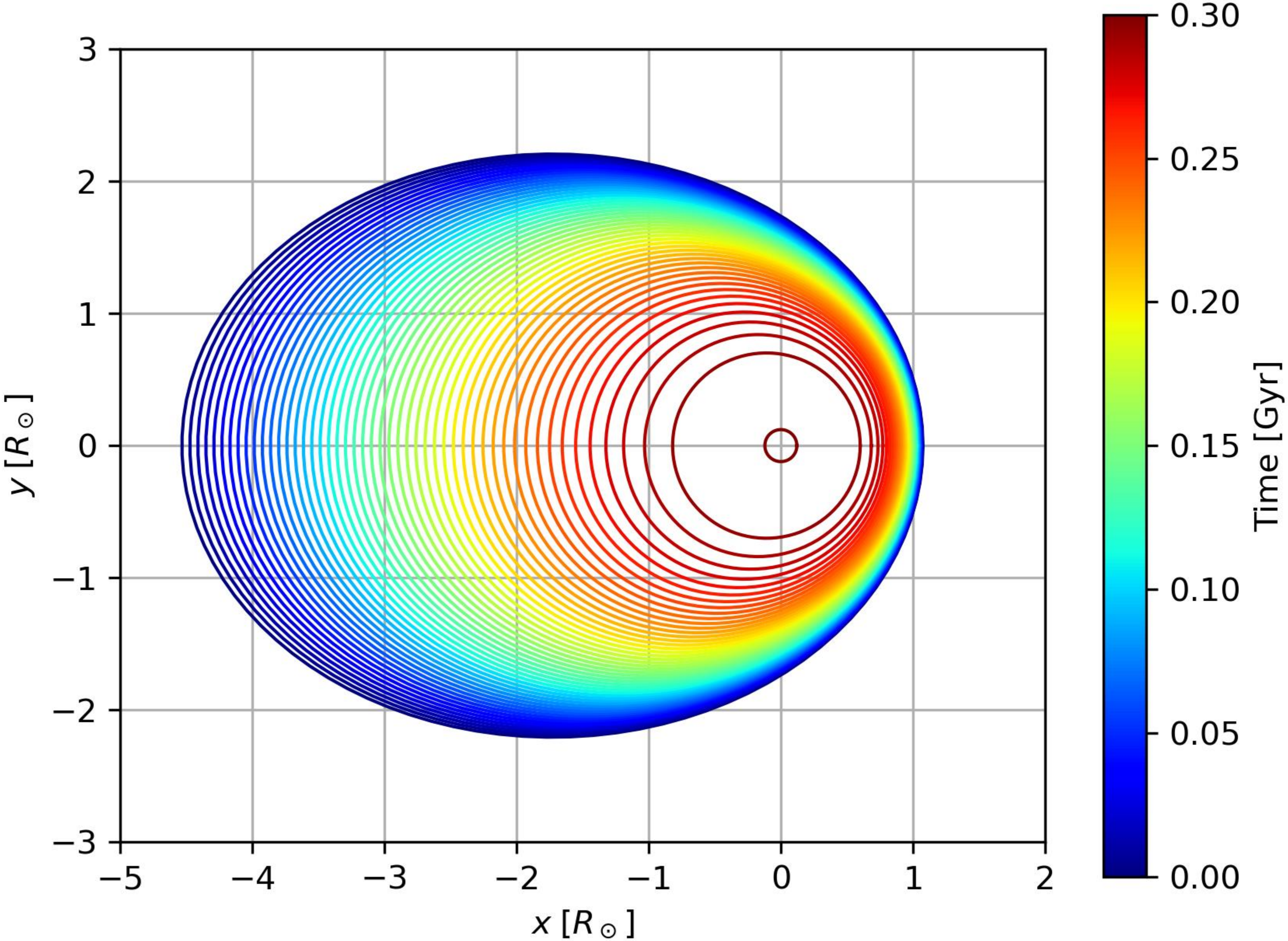


Hulse and Taylor 1974 Binary Pulsar 1913+16

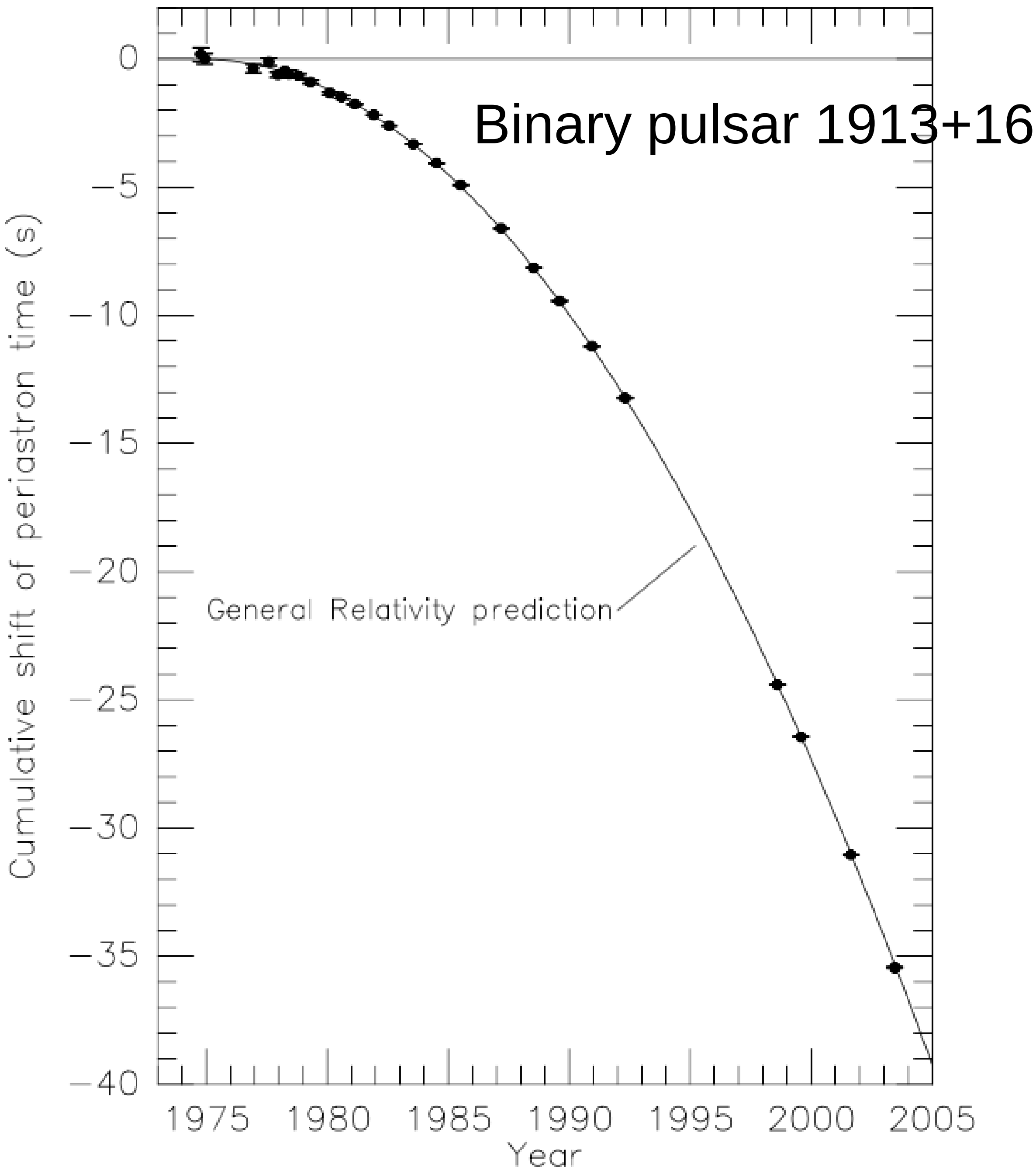


Binary stars lose energy and ang. mom. to GW!

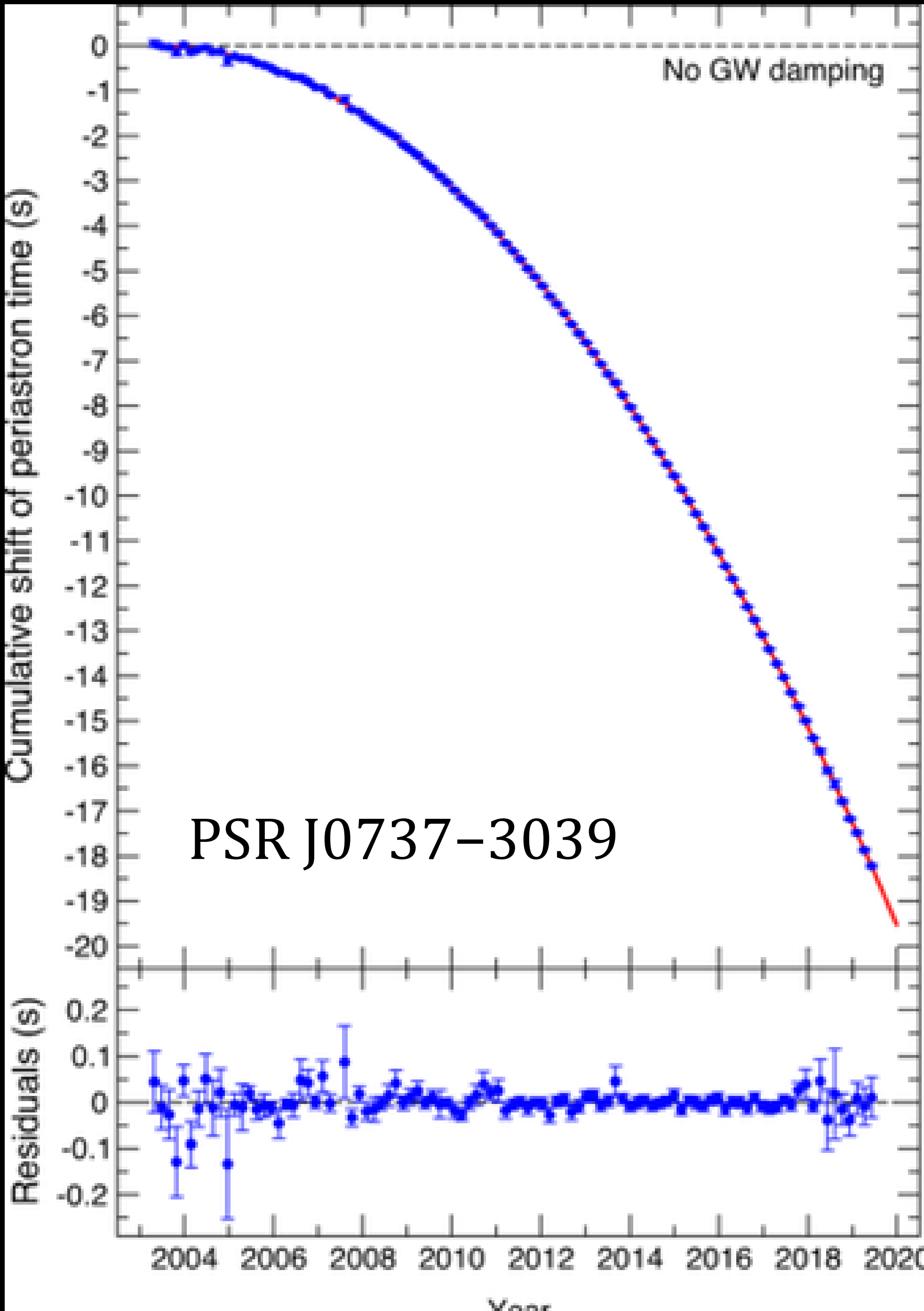

Time: 0.00 Gyr
Eccentricity: 0.62
Semi Major Axis: 2.80 $R_\odot$
y [$R_\odot$]
x [$R_\odot$]
3
2
1
0
−1
−2
−3
−5
−4
−3
−2
−1
0
1
2



3
2
1
0
−1
−2
−3
−5
−4
−3
−2
−1
0
1
2
x [R⊙]
y [R⊙]
0.30
0.25
0.20
0.15
0.10
0.05
0.00
Time [Gyr]



Binary pulsar 1913+16
General Relativity prediction
Cumulative shift of periastron time (s)
Year
0
−5
−10
−15
−20
−25
−30
−35
−40
1975
1980
1985
1990
1995
2000
2005



Discovered 2004.
Binary Pulsars,
small e = 0.088,
nearby, seen edge-on
rapid period 2.45 h.

Agreement with GR
to 0.013% !

Observation of Gravitational Waves from a Binary Black Hole Merger

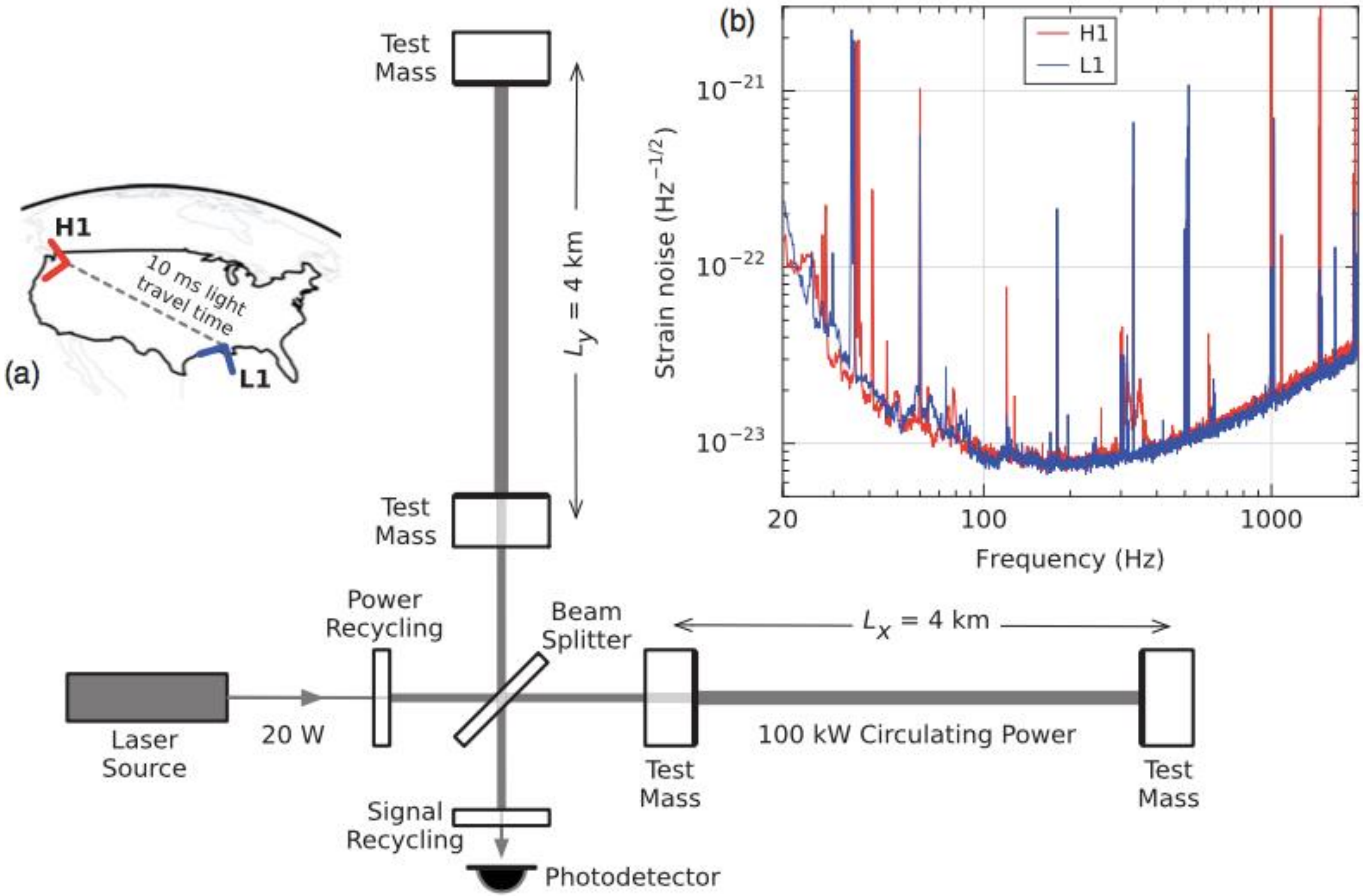
B. P. Abbott *et al.**
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ. The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_\odot$ and $29^{+4}_{-4} M_\odot$, and the final black hole mass is $62^{+4}_{-4} M_\odot$, with $3.0^{+0.5}_{-0.5} M_\odot c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

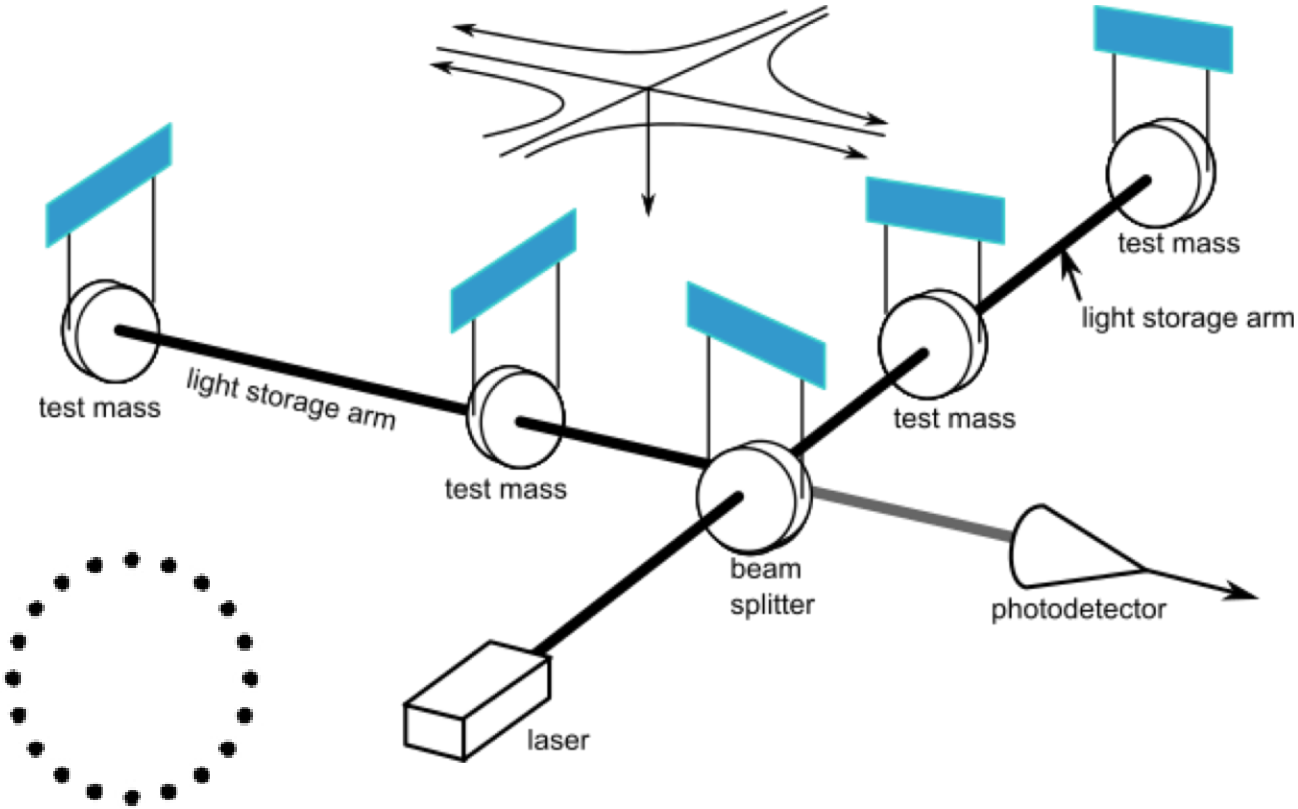
DOI: 10.1103/PhysRevLett.116.061102

The 2015 breakthrough….

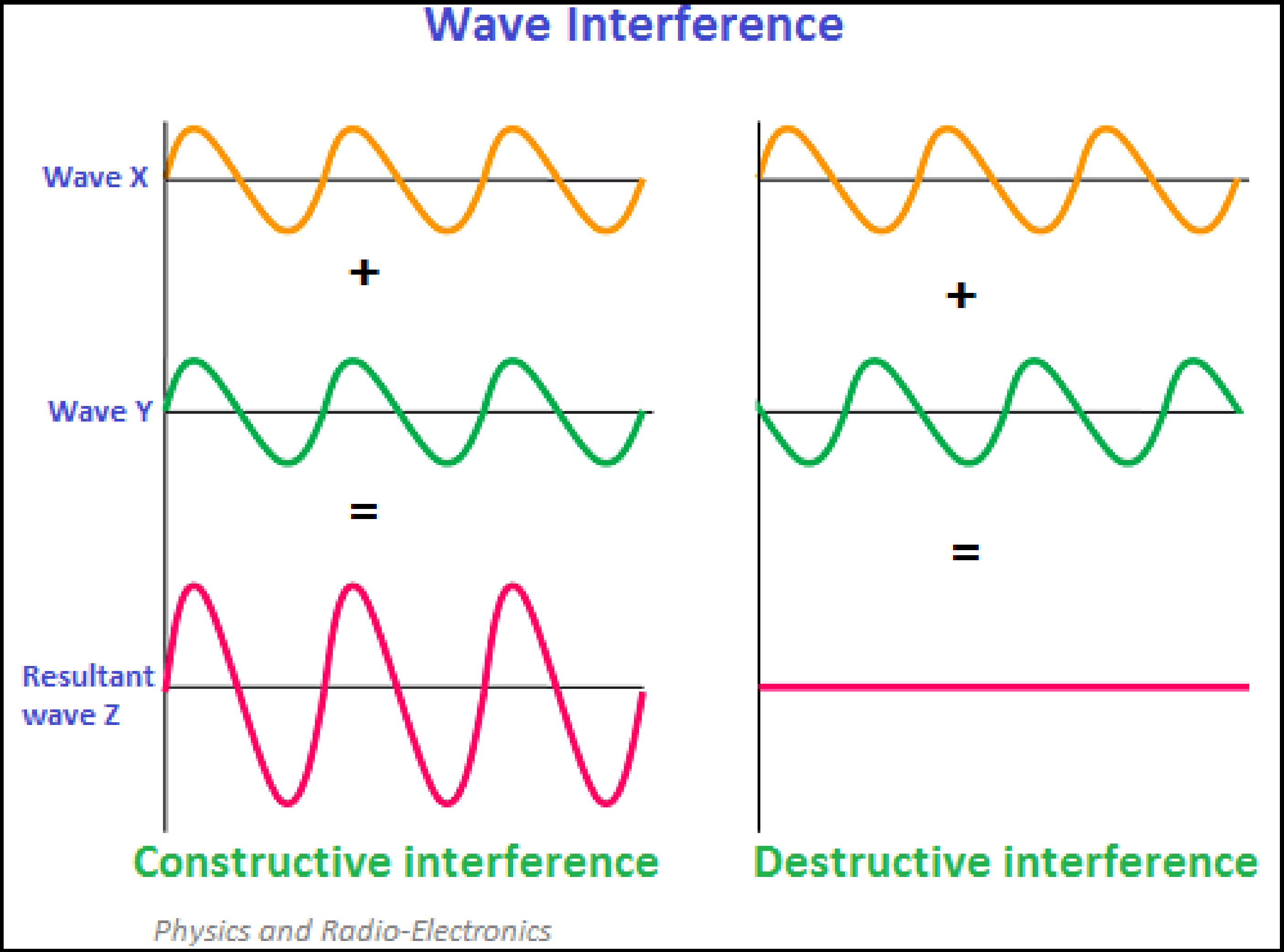
LIGO FACILITY, HANFORD WA



H1
10 ms light travel time
L1
(a)
(b)
H1
L1
Strain noise (Hz$^{-1/2}$)
10^{-21}
10^{-22}
10^{-23}
20
100
1000
Frequency (Hz)
Test Mass
Test Mass
L_y = 4 km
Power Recycling
Beam Splitter
L_x = 4 km
Laser Source
20 W
100 kW Circulating Power
Test Mass
Test Mass
Signal Recycling
Photodetector

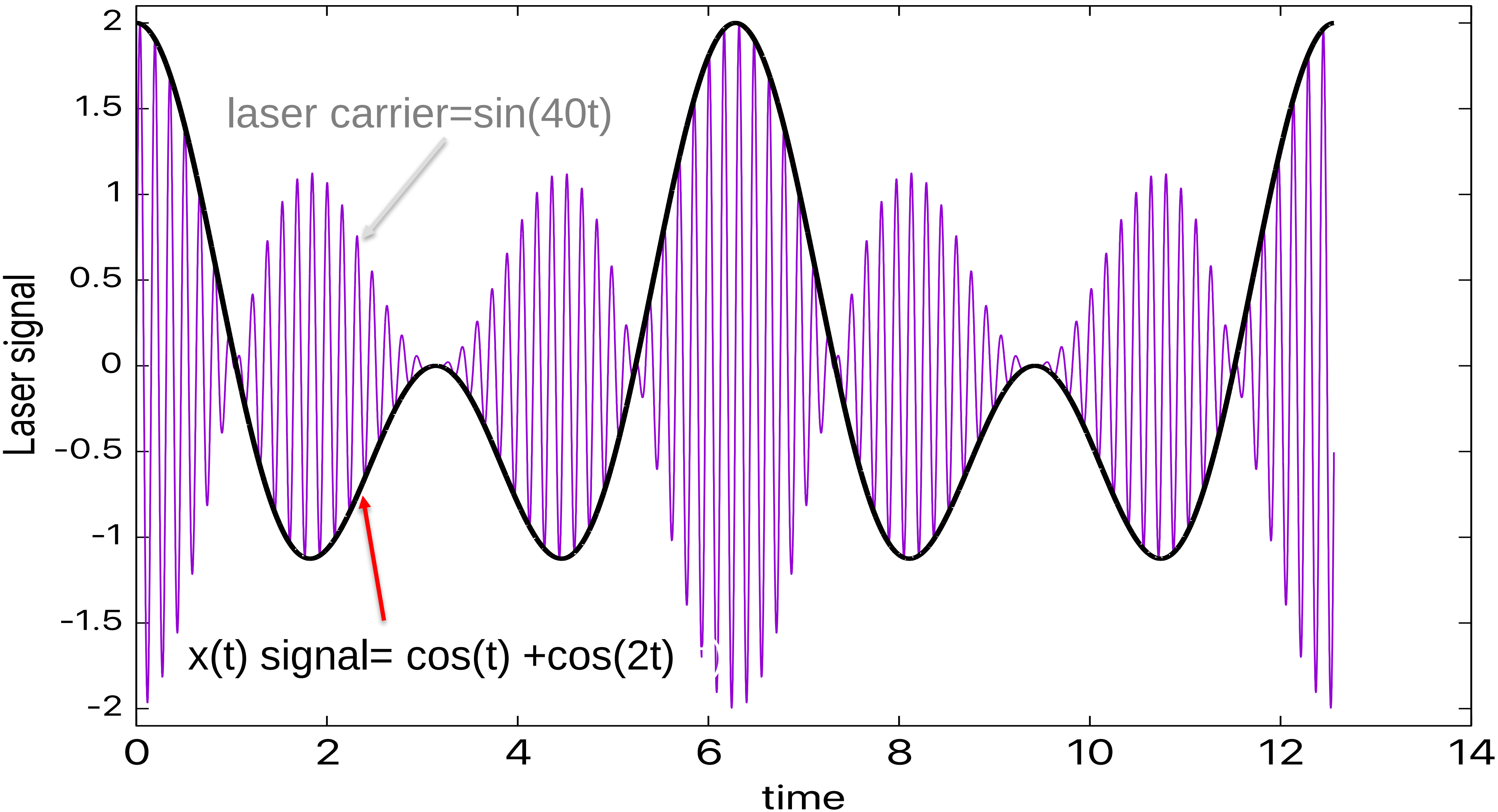


LIGO SCHEMATIC



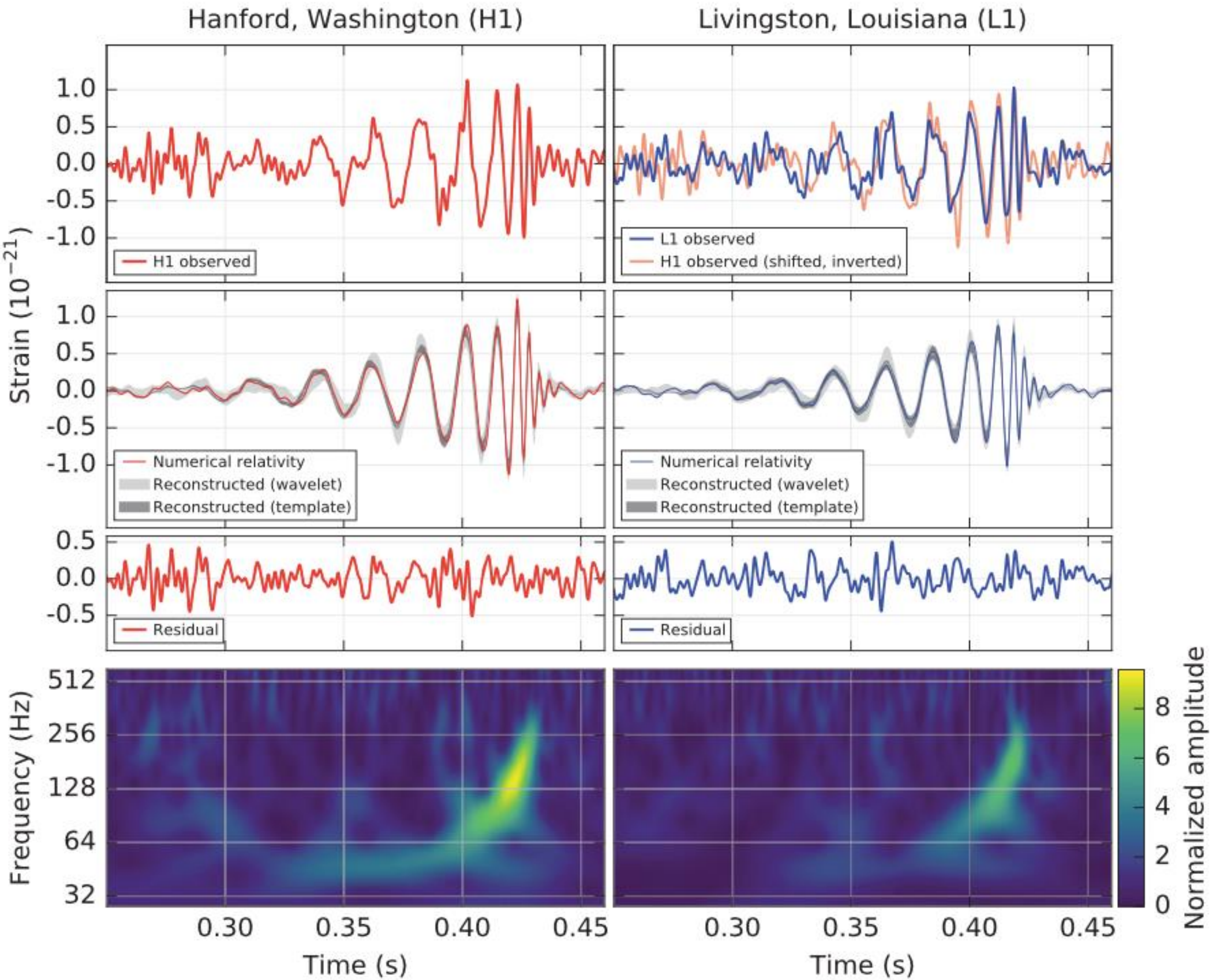
$$\cos(\omega t) + \cos(\omega t + \phi) = 2\cos(\phi/2)\cos(\omega t + \phi/2)$$

$$\cos(\omega t) + \cos(\omega t + \pi + x) \simeq x \sin \omega t$$

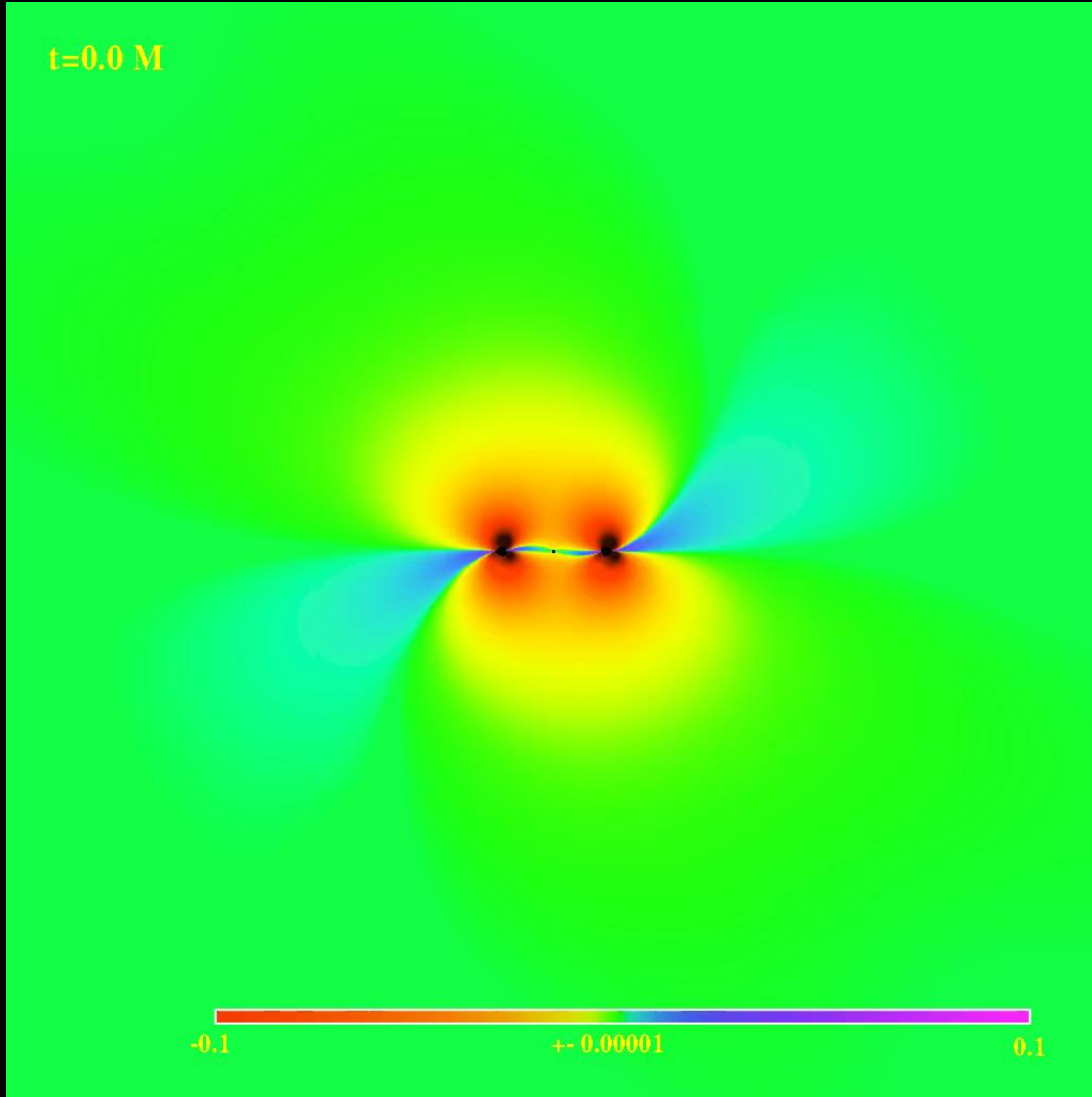


$$\cos(\omega t) + \cos(\omega t + \phi) = 2\cos(\phi/2)\cos(\omega t + \phi/2)$$

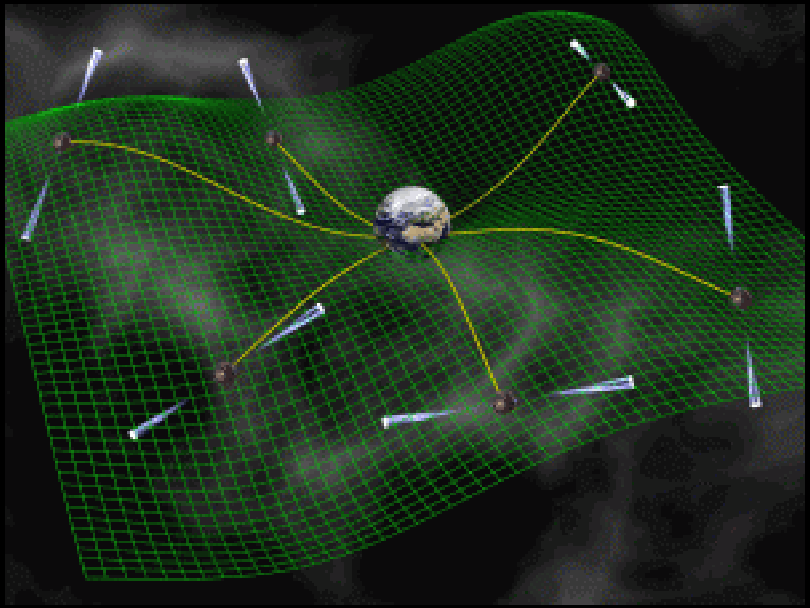
$$\cos(\omega t) + \cos(\omega t + \pi + x) \simeq x \sin \omega t$$



Hanford, Washington (H1)
Livingston, Louisiana (L1)
H1 observed
L1 observed
H1 observed (shifted, inverted)
Numerical relativity
Reconstructed (wavelet)
Reconstructed (template)
Residual
Strain (10^{-21})
Frequency (Hz)
Normalized amplitude
Time (s)

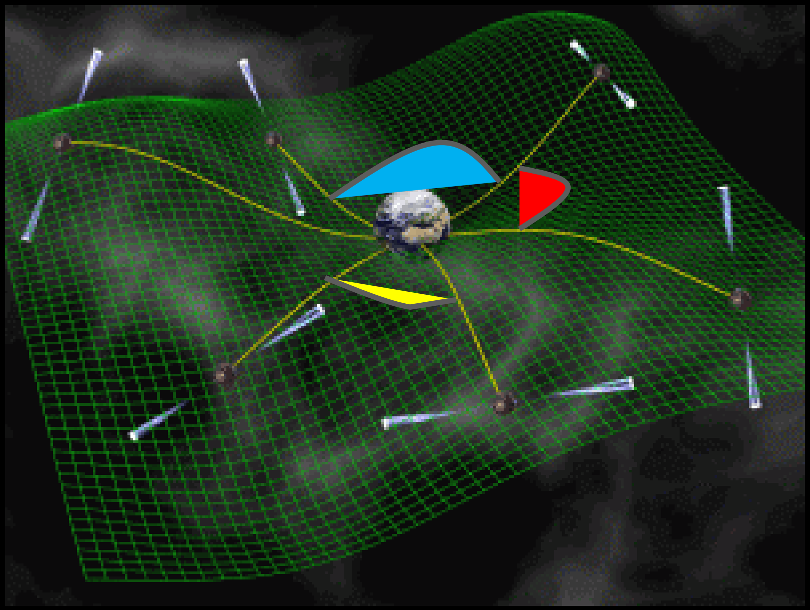

t=0.0 M
-0.1
+- 0.00001
0.1

The latest:



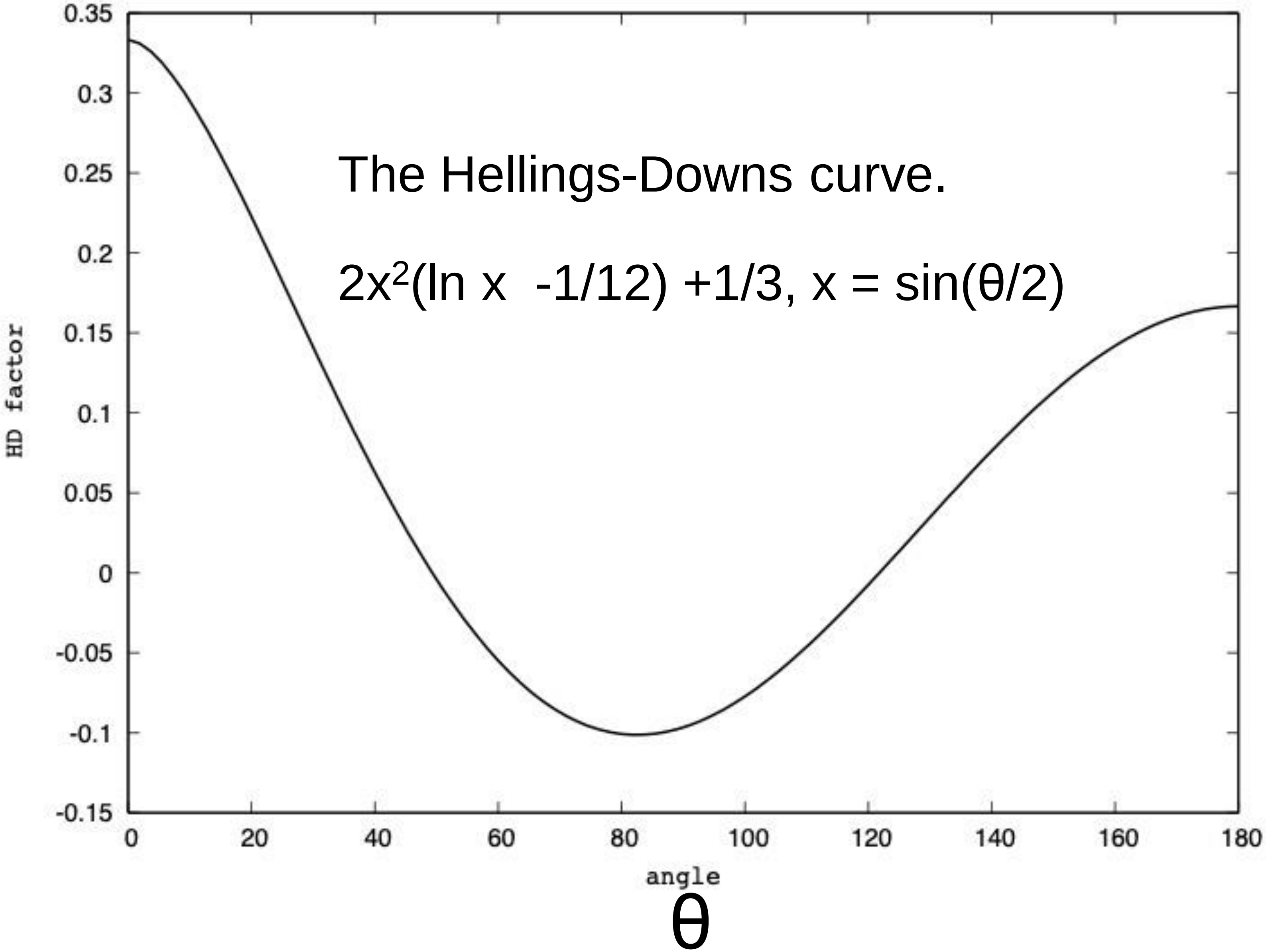
Pulsar
Timing
Array

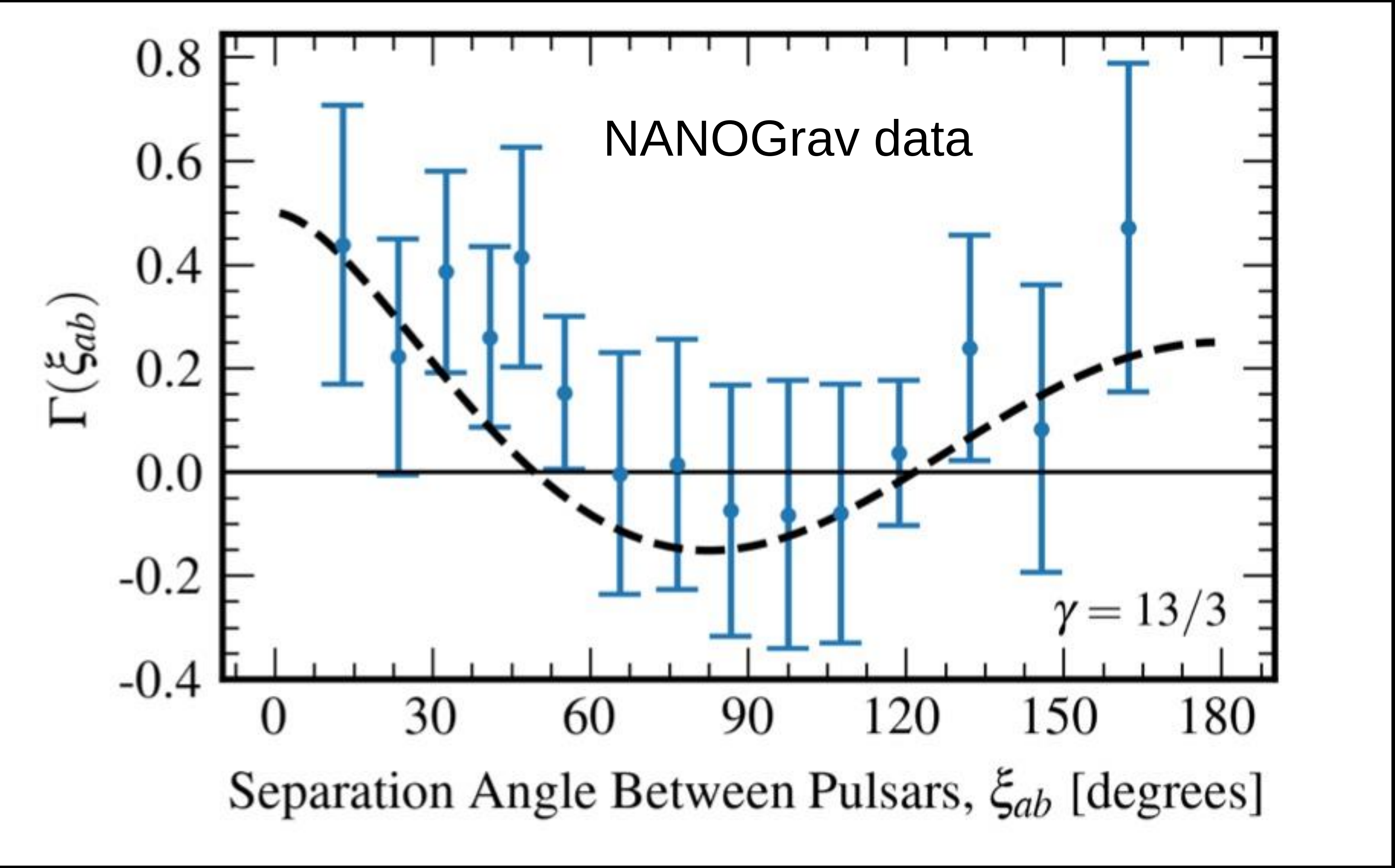
How do the changes in arrival times from two different pulsars due to the passage of a GW depend on the angular separation of the pulsar pairs?   (Time scale of years.)



The Hellings-Downs curve.

$2x^2(\ln x \; -1/12) + 1/3, \; x = \sin(\theta/2)$




NANOGrav data
$\Gamma(\xi_{ab})$
0.8
0.6
0.4
0.2
0.0
-0.2
-0.4
0
30
60
90
120
150
180
$\gamma = 13/3$
Separation Angle Between Pulsars, ξ_{ab} [degrees]

This is an unresolved background. LIGO is individual sources. The background *probably* consists of merging supermassive black holes in the centres of galaxies throughout the Universe. We may soon see some individual sources poke out of the background….and we may also, of course, be taken by surprise!